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Alternating Minimax Approximation with Unequal Restraints

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1. INTRODUCTION

For $g \in C[a, b]$ define

$$\| g \| = \sup \{| g(x)|: a \leqslant x \leqslant b\}.$$

Let F be an approximating function with parameter such that P is the parameter space and $F(A, \cdot) \in C[a, b]$ for all $A \in P$. Let u, v be continuous mappings into the extended real line, $u < v$. The approximation problem is: for a given $f \in C[a, b]$, $u \leqslant f \leqslant v$, to find $A^* \in P$ satisfying the restraint

$$u \leqslant F(A^*, \cdot) \leqslant v \tag{1}$$

for which $e(A) = \| f - F(A, \cdot)\|$ is minimal. The parameter A^* is called best to f and $F(A^*, \cdot)$ is called a best restrained minimax approximation to f.

The case $u = -\infty$, $v = \infty$ corresponds to Chebyshev approximation. The cases $u = -\infty$, $v = f$ and $u = f$, $v = \infty$ correspond to one-sided approximation. The case $u = 0$, $v = \infty$ is that of nonnegative approximation of nonnegative functions.

In [7, p. 72] the related problem of interpolation with restraints was studied.

The dissertation [8] studied a problem less general than that of this note, but included results on approximation with respect to a weight function and on the continuity of the best approximation operator.

2. ALTERNATING FAMILIES

We will be solely concerned with the case in which (F, P) is an alternating family on $[a, b]$, that is, F has a degree $\rho(A) > 0$ at all parameters A (or, equivalently, $F(A, \cdot)$ is best to f on $[a, b]$ if and only if $f - F(A, \cdot)$ alternates

$\rho(A)$ times on $[a, b]$). For details see [1, p. 225] or [3, pp. 17–22]. Examples include families of power polynomials, polynomial rational families, unisolvent families, and some families of exponential functions.

DEFINITION. F has property Z of degree n at A if $F(A, \cdot) - F(B, \cdot)$ having n zeros implies $F(A, \cdot) \equiv F(B, \cdot)$. Double zeros (defined later) are not counted twice.

DEFINITION. F has property $\mathcal{A}$ of degree n at A, if for any integer $m < n$, any sequence $\{x_1, ..., x_m\}$ with

$$a = x_0 < x_1 < \cdots < x_{m+1} = b,$$

any sign σ, and any real ϵ with

$$0 < \epsilon < \min\{x_{j+1} - x_j : j = 0, ..., m\},$$

there exists a $B \in P$, such that

$$\| F(A, \cdot) - F(B, \cdot)\| < \epsilon,$$

$$\begin{aligned} \operatorname{sgn}(F(A, x) - F(B, x)) &= \sigma, & a \leqslant x \leqslant x_1 - \epsilon \\ &= \sigma(-1)^j, & x_j + \epsilon \leqslant x \leqslant x_{j+1} - \epsilon \\ &= \sigma(-1)^m, & x_m + \epsilon \leqslant x \leqslant b. \end{aligned}$$

In case $m = 0$, the above sign condition reduces to

$$\operatorname{sgn}(F(A, \cdot) - F(B, \cdot)) = \sigma.$$

DEFINITION. F has degree n at A if F has property Z of degree n at A and property $\mathcal{A}$ of degree n at A. Denote this degree by $\rho(A)$.

DEFINITION. A point x in (a, b) such that $g(x) = 0$ but g does not change sign is called a *double zero* of g.

LEMMA 1. *Let F have positive degree at A and B. If $F(A, \cdot) - F(B, \cdot)$ has $\rho(A)$ zeros, counting double zeros twice, then $F(A, \cdot) \equiv F(B, \cdot)$.*

This lemma first appeared in [1, p. 225] without a detailed proof. A generalization with a complete proof appears in [2, Lemma 7].

3. Characterization of Best Approximation

To give added generality we will let u be upper semicontinuous into the extended real line $\bar{R}$ and v be lower semicontinuous into $\bar{R}$ (for definitions see [6]). It follows that $F(A, \cdot) - u$ is lower semicontinuous into $\bar{R}$ and so attains its infimum on a closed set. Similarly, $v - F(A, \cdot)$ is lower semicontinuous into $\bar{R}$ and so attains its infimum on a closed set.

Definition. x is a *minus point* of $f - F(A, \cdot)$ if $f(x) - F(A, x) = -e(A)$ or $F(A, x) = v(x)$.

Definition. x is a *plus point* of $f - F(A, \cdot)$ if $f(x) - F(A, x) = e(A)$ or $F(A, x) = u(x)$.

By continuity of $f - F(A, \cdot)$ and lower semicontinuity of $F(A, \cdot) - u$, it follows that for $F(A, \cdot) \geqslant u$, the set of plus points is closed. Similarly, for $F(A, \cdot) \leqslant v$, the set of minus points is closed. There is no point which is both a minus point and a plus point unless $e(A) = 0$. Suppose, for example, we have $f(x) - F(A, x) = -e(A)$ and $F(A, x) = u(x)$, then $f(x) - u(x) = -e(A)$. As f satisfies $f \geqslant u$ we can only have $e(A) = 0$. By continuity of $|f - F(A, \cdot)|$ there is a point x with $|f(x) - F(A, x)| = e(A)$.

Definition. $f - F(A, \cdot)$ is said to *alternate* n times with respect to u, v if there is a set $\{x_0, ..., x_n\}$, $a \leqslant x_0 < \cdots < x_n \leqslant b$, such that the points are alternately plus points and minus points. The set is called an *alternant*.

Before characterizing best approximations in terms of alternation, we develop a de la Vallée–Poussin type result which characterizes near-best approximations.

Definition. x is a weak minus point of $f - F(A, \cdot)$ if $f(x) - F(A, x) < 0$ or $F(A, x) = v(x)$. x is a weak plus point of $f - F(A, \cdot)$ if $f(x) - F(A, x) > 0$ or $F(A, x) = u(x)$.

Lemma 2. *Let A satisfy* (1). *Let $\rho(A) = n$ and $x_0 < x_1 < \cdots < x_n$ be alternately weak plus points and weak minus points of $f - F(A, \cdot)$. Then for any parameter B for which* (1) *is satisfied and at which F has a degree, $F(B, \cdot) \not\equiv F(A, \cdot)$,*

$$\max\{|f(x_i) - F(B, x_i)| : i = 0, ..., n\}$$
$$> \min\{|f(x_i) - F(A, x_i)| : i = 0, ..., n, F(A, x_i) \neq u(x_i), F(A, x_i) \neq v(x_i)\}$$

Proof. Suppose not. Assume without loss of generality that x_0 is a weak plus point, then we have

$$(-1)^i[F(B, x_i) - F(A, x_i)] \geqslant 0, \qquad i = 0,..., n, \tag{2}$$

and $F(A, \cdot) - F(B, \cdot)$ must have n zeros counting double zeros twice. By Lemma 1, $F(A, \cdot) \equiv F(B, \cdot)$.

Note. In the case $F(A, x_i)$ is alternately $u(x_i)$ and $v(x_i)$, the right-hand side in the lemma is undefined. Assume without loss of generality that $F(A, x_0) = u(x_0)$, then for B satisfying (1) we have (2) and it follows that $F(B, \cdot) \equiv F(A, \cdot)$, that is, there is only one acceptable approximation.

LEMMA 3. *Let F have a positive degree at all parameters and $\rho(A) = n$. Let $f - F(A, \cdot)$ alternate n times and A satisfy* (1), *then A is best.*

Proof. Let $\{x_0, ..., x_n\}$ be an alternant. In the case $F(A, x_i)$ is alternately $u(x_i)$ and $v(x_i)$, $F(A, \cdot)$ is the only acceptable approximant by the note above. If this is not the case then there exists j such that $F(A, x_j) \neq u(x_j)$, $F(A, x_j) \neq v(x_j)$, hence $|f(x_j) - F(A, x_j)| = e(A)$. By Lemma 2, if B satisfies (1), $\rho(B) > 0$ and $F(B, \cdot) \not\equiv F(A, \cdot)$,

$$e(B) \geqslant \max\{|f(x_i) - F(B, x_i)| : i = 0,..., n\} > e(A).$$

THEOREM. *Let F have a positive degree at all parameters. A necessary and sufficient condition for A satisfying* (1) *to be a best approximation is that $f - F(A, \cdot)$ alternate $\rho(A)$ times with respect to u, v.*

Proof. Sufficiency follows from Lemma 3. We now prove necessity. Suppose $f - F(A, \cdot)$ has no alternations. Assume without loss of generality that $f - F(A, \cdot)$ has a plus point. Let $M = \inf\{f(x) - F(A, x): a \leqslant x \leqslant b\}$. If $M = -e(A)$ then there exists x such that $f(x) - F(A, x) = -e(A)$ and x is a minus point. We would then have a plus point and a minus point, hence at least one alternation, which is contrary to hypothesis. Let $\delta = M + e(A)$, then $\delta > 0$. There is no point y such that $F(A, y) = v(y)$ for such a point would be a minus point, which would give alternation. As $v - F(A, \cdot)$ is lower semicontinuous, it attains its infimum η which is therefore positive. Let $\epsilon = \min\{\delta, \eta\}/2$ and by property $\mathcal{Q}$ choose $B \in P$ such that

$$F(A, \cdot) < F(B, \cdot) < F(A, \cdot) + \epsilon.$$

As $u \leqslant F(A, \cdot)$ we have $u < F(B, \cdot)$ and as $F(A, \cdot) + \epsilon < v$, we have $F(B, \cdot) < v$, hence B satisfies (1). Further,

$$\begin{aligned} -e(A) &\leqslant f - F(A, \cdot) - \delta < f - F(A, \cdot) - \epsilon < f - F(B, \cdot) \\ &< f - F(A, \cdot) \leqslant e(A). \end{aligned}$$

Next consider the case where $f - F(A, \cdot)$ alternates exactly m times, $0 < m < \rho(A)$. We can divide $[a, b]$ into $m + 1$ subintervals I_k, $k = 0,..., m$, such that none contains both minus points and plus points, and no interior endpoint of the subintervals is a plus or minus point. Let J_k be a closed interval in I_k containing the plus or minus points which are not endpoints of $[a, b]$ in its interior. Assume without loss of generality that I_0 contains plus points. Define

$$M_k = \inf\{(f(x) - F(A, x))(-1)^k : x \in J_k\}.$$

As J_k is closed and contains no minus (plus) points for k even (odd), $M_k > -e(A)$. Define

$$\delta = \min\{M_k : k = 0,..., m\} + e(A),$$

then $\delta > 0$ and

$$\begin{aligned} f(x) - F(A, x) - \delta &\geqslant -e(A), \qquad x \in J_k, \quad k \text{ even}, \\ f(x) - F(A, x) + \delta &\leqslant e(A), \qquad x \in J_k, \quad k \text{ odd}. \end{aligned}$$

Let k be even. There is no point $x \in J_k$ such that $F(A, x) = v(x)$, for such a point would be a minus point. As $v - F(A, \cdot)$ attains its infimum on closed J_k, it follows that there exists $\mu_k > 0$ such that

$$v(x) - F(A, x) \geqslant \mu_k, \qquad x \in J_k, \quad k \text{ even}.$$

A similar argument shows that for k odd, there exists $\mu_k > 0$ such that

$$F(A, x) - u(x) \geqslant \mu_k, \qquad x \in J_k, \quad k \text{ odd}.$$

Define $\mu = \min\{\mu_k : k = 0,..., m\}$. Let $K = [a, b] \sim \bigcup_{k=0}^{m} J_k$. Define $\rho = \sup\{|f(x) - F(A, x)| : x \in \overline{K}\}$. As $\overline{K}$ has no plus or minus points and is closed, $\rho < e(A)$.

Define

$$L = \inf\{\inf\{v(x) - F(A, x), F(A, x) - u(x)\} : x \in \overline{K}\}.$$

As $v - F(A, \cdot)$, $F(A, \cdot) - u$ are lower semicontinuous, L is attained on $\overline{K}$ and $L > 0$. Let $\epsilon = \min\{\delta, \mu, L, e(A) - \rho\}/2$. By property $\mathcal{A}$ of degree $\rho(A)$ at A, we can choose $B \in P$ such that $\| F(A, \cdot) - F(B, \cdot)\| < \epsilon$ and

$$\operatorname{sgn}(F(B, x) - F(A, x)) = (-1)^k, \qquad x \in J_k.$$

For $x \in J_k$, k even, we have

$$\begin{aligned} u(x) &\leqslant F(A, x) < F(B, x) < F(A, x) + \epsilon < F(A, x) + \mu_k \leqslant v(x), \\ -e(A) &\leqslant f(x) - F(A, x) - \delta \leqslant f(x) - F(A, x) - \epsilon < f(x) - F(B, x) \\ &< f(x) - F(A, x) \leqslant e(A). \end{aligned}$$

For $x \in J_k$, k odd, we have

$$\begin{aligned} u(x) &\leqslant F(A, x) - \mu_k < F(A, x) - \epsilon < F(B, x) < F(A, x) \leqslant v(x), \\ -e(A) &\leqslant f(x) - F(A, x) < f(x) - F(B, x) < f(x) - F(A, x) + \epsilon \\ &< f(x) - F(A, x) + \delta \leqslant e(A). \end{aligned}$$

Let $x \in K$, then

$$\begin{aligned} |f(x) - F(B, x)| &\leqslant |f(x) - F(A, x)| + |F(A, x) - F(B, x)| \\ &\leqslant \rho + \epsilon \leqslant \rho + (e(A) - \rho)/2 = (e(A) + \rho)/2 < e(A), \\ v(x) &\geqslant F(A, x) + L > F(B, x) - \epsilon + L > F(B, x), \\ u(x) &\leqslant F(A, x) - L < F(B, x) + \epsilon - L < F(B, x). \end{aligned}$$

Combining the inequalities for x in J_k (k even), in J_k (k odd), and K, we have

$$u < F(B, \cdot) < v,$$
$$-e(A) < f - F(B, \cdot) < e(A).$$

Hence $F(B, \cdot)$ is a better approximation and necessity is proven.

COROLLARY. *A best approximation to f is unique.*

Proof. By the theorem a best approximation $F(A, \cdot)$ must have an alternant of length $\rho(A) + 1$. We apply Lemma 2 to get $e(B) > e(A)$ if $F(B, \cdot) \not\equiv F(A, \cdot)$.

The case where u may equal v at some points is more complex. Some cases in polynomial approximation are given in [5]. It is possible for u and v to agree at only one point and only one approximation exists satisfying (1).

EXAMPLE. Let $[a, b] = [0, 1]$ and the approximating family be all power polynomials of degree n. Let $u(x) = -x^{n+1}$, $v(x) = x^{n+1}$, then the only approximant which lies between u and v is the zero approximant.

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